New perspectives for Geometry teaching: Mechanical linkages Technology

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Abstract: In this paper we try a brief presentation of mechanical linkages, and especially of the drawing machines. Our focus is on the pantograph, that incorporates mathematical properties and relationships in structure in such a way to allow the implementation one geometrical transformation, such as, symmetry, reflection, translation and homothety. In order to investigate subjects’ concepts/ theorems-in-action developed by investigating the structure of the pantograph, and especially the identification of the math concepts and laws incorporated in the machine, we selected a pantograph’s model and taught homothety to high school students for four hours (early 2016), in the framework of an attempt to incorporate artifacts with the characteristics geometrical machine’s in the instruction of Euclidean geometry.

Keywords: ‘linkages’, ‘pantograph’, ‘geometry’, ‘teaching’, ‘homothety’

1. Introduction

Learning with the use of various technological tools attracts a lot of research interest and holds great promise; the use of the tools in the current learning environments allows the students to have increased chances of achieving mathematical concepts, exploring and experimenting with mathematical ideas and expressing such ideas and concepts via a variety of representations (Tamina, 2008). At the same time, integrating engineering-based problem solving within the learning objectives of the students is also very appealing in the progressive societies, making the teaching of Science, Technology, Engineering and Mathematics (STEM) a very important component of their education.

Technology is often referred to as the tool and the application of math and science. Within the classroom, computer technology is often used to facilitate lesson planning, class activities or even to create resources. Studying the history of technology, during the mechanical age (between 1450 and 1840) a lot of new technologies were developed. These technologies embodied the knowledge of their time, and were the first attempt of math and science modeling. Such technologies include the slide rule (an analog computer used for multiplying and dividing) and the Pascaline (a very popular mechanical computer) by Blaise Pascal that could be used even nowadays. Nevertheless, the use of mathematical machines as technological tools to promote learning seems to be under-researched, with limited existing knowledge regarding the experience of the students. A sound exception is the MMLab (Laboratory of Mathematical Machines, www.mmlab.unimore.it), where researchers (Bartolini Bussi, 2010; Martignone, 2011; Mariotti et al., 1997) have investigated from an epistemological and pedagogical aspect the use of mechanical machines -concerning geometry and arithmetic - as a way to generate mathematical ideas or concepts in the classroom. This Research Group has investigated the use of simple mathematical machines in different contexts and grade
levels. These machines (for example pantographs) are linkages that allow the implementation of geometrical transformations, such as symmetry, reflection, translation, and homothety. Unlike some artifacts from ICT (Information Computer Technology), they need to be handled appropriately, require motor abilities, might resist the motion and need time to be explored. The role of mathematical machines notably those that are part of the historical phenomenology of geometry, as a rule, compasses, mechanisms and geometrical machines (for example the pantographs), and their didactic exploitation in activities are the subjects of research studies and their use is discussed in teaching activities in the classroom (Mariotti, 2002; Maschietto, 2005; Maschietto & Bussi, 2011).

The use of simple machines was under investigation also in the SIMALE project (the Simple Machines Learning Environment) aimed to support the mechanical reasoning and understanding of middle and high school students. The project was created to support “reflection, collaboration, and presentation of concepts from multiple perspectives and contexts” (McKenna & Agogino 2004, p. 97) and resulted in the development of a learning environment.

In this paper, we focus on geometrical mechanical linkages in order to explore their nature and their role in geometry teaching. More specifically, we focus on the pantograph as an example of didactic exploration of a simple machine model in classroom activities about similarity.

**How different technologies historically have affected geometrical thinking?**

2. Machines and Mechanisms

One of the most fundamental examples of machines are the linkages. A mechanical linkage is a series of rigid links connected with joints to form a closed chain, or a series of closed chains by having a link or links fixed, and by means of which straight or nearly straight lines or other point paths may be traced. Each link has two or more joints, and the joints have various degrees of freedom to allow relative movement. Mechanical linkages are usually designed to take an input and produce a different output, altering motion, velocity, acceleration, and applying mechanical advantage. If two or more links are movable with respect to a fixed link then a linkage is called a mechanism.

![Figure 1: Designs of Linkages to generating Straight-line Motion](image)

Linkage design is often divided into three categories of tasks called *motion generation, function generation* and *point-path generation*, respectively (McCarthy, 2006). The point-path generation category is a classical problem in linkage design where the primary concern is the generation of straight-line paths, some examples shown in Figure 1. One of the simplest examples of linkage is the four-link mechanism or four-bar linkage mechanism. A variety of useful mechanisms can be formed from a four-link mechanism (as in Figure 2, with slight variations, such as changing the character of the pairs, proportions of links, etc.)
Mechanical linkages found great applicability during the Industrial Revolution. The advances in the fields of mathematics, engineering and manufacturing processes set the ground and the need to create new mechanisms; what nowadays appears to be a simple and obvious mechanism, requires the contribution of some of the greatest minds of the time such as Leonhard Euler, James Watt (as in Figure 1a) and Pafnuty Chebyshev (as in Figure 1c).

2.1. Mechanisms Designed by Geometric Structure

According to Taimina (2008) “the main aspects of geometry today emerged from four strands of early human activity that seem to have occurred in most cultures: art/pattern, building structures, navigation/stargazing, motion in machines” and “these strands developed more or less independently into varying studies and practices that eventually from the 19th century on were woven into what now call geometry” (Taimina & Henderson, 2005).

Geometry and motion came close together for ancient engineers and mechanics has been developed along with mathematics. In ancient Greece, Archimedes, Heron and other geometers used mechanisms and gears to solve geometrical problems; trisecting an angle, duplicating a cube and squaring a circle are problems that could easily be solved with the use of motion. One of the first mechanical solutions was offered by Menaechmus; he constructed a mechanical device that would trace two appropriate conic curves. Plato criticized this mechanistic approach and called instead for a purely theoretical solution with geometrical tools, as the rule and the compasses. Descartes (1596-1650), in his ‘Geometry’ (1637), discussed that it is obscure for him that the ancients wanted to discuss geometric construction only using circle and line and provided a response to the question 'how to make geometric constructions that they cannot be constructed with ruler and compass’. He changed the roles of ruler and compasses from exclusive geometrical tool for geometry to one of mechanical tools which were used for representing mathematics and his view enhanced exploration of mechanics and influenced the technological and mechanical sciences and the cognition (Isoda, & Matsuzaki, 1999).

2.2. Special linkages: The Drawing Machines

Before the computer era, mathematicians tried to design physical mechanics and some kinds of mechanics like figure 1, 2 and 3 still have roles as geometrical tools or drawing machines whose the mechanical structure is represented by a geometric structure, such as for example, the devices of Figures 4 and 5.

Figure 2: Four-Bar Linkage Mechanism (Wikipedia)
**Drawing machine** is any device/apparatus/mechanism/instrument that draws or assists in the act of drawing. Since the early 1400s, scientists and inventors have created devices to assist in drawing. And this is not limited to "drawing realistically from life." Prisms attached to microscopes, gears and linkages joining forces for complex geometrical drawings, and intricate, specialized drafting tools are also drawing machines.

![Figure 6: Linkages for Machine and Pantograph for drawing from Secondary Mathematics Japanese Textbook in 1943 (approved by Monbusyo), Isoda (1997)](image)

Referring to a virtual mechanism, (the mechanism model) Liu et al. (2012, p.1105) distinguish between two parts of the mechanism model. (1) The information model which is used to describe the data structure of mechanism and provide information for mathematical model analysis, and (2) The mathematical model used to describe the mathematical expression of mechanism. The information model describes the data structure of mechanism. Mechanism is composed by two parts: links and kinematic pairs, so the information model of mechanism should contain these two basic data structure. The links of the mechanism are connected through the kinematic pairs, so the motion of the links are limited in order to make the mechanism move along the route which is designed by the operator. So the mathematical model of mechanism should be used to describe the constraint relationships among the links which belong to the mechanism (p.1108). For example, the movement of a machine can be described by the geometric representation as a locus that being 'written' (as a kinematic geometry of movement). Even in mechanics, one can select between the many mathematical representations; in some cases real situations can be more accurately described by geometrical representation and not algebraic ones. By choosing appropriate link lengths and coupler point locations, useful curves can be found, which are formalized by applying geometry to the analysis and synthesis of machines. Reuleaux’s theories about machines, based largely on geometric ideas and not on dynamic requires information on the changing geometry during motion. The ground rules: a) a drawing machine must control—or help a user control—a stylus, b) when used to draw from life, a drawing machine inserts itself into the stylus-hand-eye circuit, and c) when functioning as a plotter, a drawing machine can be an autonomous or semi-autonomous machine. (Martinez et al., 2010)

### 2.3. Special Drawing Machines: The Pantographs

The pantograph is a special drawing machine. This is an articulate system of rods which are connected with links forming simple geometrical shapes, such as similar or equal triangles, parallelograms and rhombus. The pantograph is a geometrical machine, a tool that forces a point to follow a trajectory or to be transformed according to a given law embedded in the structure of the machine (Bussi & Maschietto, 2008) and—that incorporates mathematical properties and relationships in structure in such a way to allow the implementation one geometrical transformation, such as, symmetry, reflection, translation and homothetic e.g. Some kinds of pantographs’ shown in Figure 7. The links allow the rods to rotate, which makes the tool not static in form and to the magnitude. The hinge/joint allows the transformation of the form of the machine and the change of the angles, without making changes in lengths and the relationships between them. The drawing-
point writes locus that the proof may became within Euclidean geometry as well as that of analytic geometry. This is a good example to in order the students know the difference between of the two systems in mathematics (Bussi, et al., 2010).

**Axial symmetry**  **Central Reflection**  **Relocation**  **Translation**  **Homothetic**

![Figure 7: Pantographs’ Kinds (http://www.mmlab.unimore.it)](http://www.mmlab.unimore.it)

### 2.4. A Special Pantograph for homothety

The idea of a pantograph is based on geometrical proportions that were known since ancient times. This drawing instrument is used for copying drawings and maps to a different scale, or in a different role for guiding a cutting tool in a manufacturing process. Gabriel Koenigs (1858–1931) expressed well the complex links between pure and applied mathematics, referring to a pantograph for homothety. “The theory of linkages is supposed to start in 1864. Surely linkages were used also earlier: a dedicated and precise scholar might track down them in the most ancient times. One might discover in this way that each age has in hand, so to say, yet without awareness, the discoveries of future ages: the history of things often anticipates the history of ideas. When in 1631 Scheiner published for the first time the description of his pantograph, he certainly did not know the general concepts contained as germs in his small instrument; we claim that he could not know them, as they are linked to the theory of geometric transformations, that is a theory typical of our century and gives a unitary stamp to all the made advances” (Koenigs, 1897, in Bussi, et al., 2010, p.26).

In the case of the pantograph, like figures 2, 3 and 5 similarity is kept by the geometric structure which is based on the parallelogram and the line through the fulcrum, the force point and the influence point. In the shapes of the Figures 5, 6 (Bolt, 1991) above the quadrilateral AYBC is a rhombus and AX=BZ and both are equal to the side of the rhombus. The result is that no matter how the linkage is moved X, Y and Z will always be in a straight line. If the point X is fixed (fixed-point), the point Y traces around an object (traced-point), then the point Z (drawing-point) make a copy enlargement of the object with scale factor equal ZX/YX. For example, the following schematics show that, in Figure 5 the pantograph has been recommended to produce an enlargement with scale factor equal 2 because Z is always twice as far from X as Y is, and the Figure 6 the scale factor of the enlargement is 4, because X, Y, Z stay in a straight line and YZ=3XY, so making ZX=4YX. Isoda & Matsuzaki (1999, p. 115) point out that “if students know the structure of the pantograph, they must change the ratio of similarity as parameters. If not, they could explore the pantograph by changing parts and discover the conditions which keep the similarity. Cognitive structures must be strictly difference before and after knowing the structure. Before knowing the structure they enjoyed changing parts but after knowing the structure there were parameters which should be changed”. They note that “we should also know that same mechanics can be manipulated according to one’s intuition which depends upon user’s knowledge of mathematics. It should not forget that mechanics helps to develop mathematics. For example, Descartes’ intuition about curves could not easily sheared with us without using his instrument”. The manner of handling and use of the mechanism depends on the intuitions of the person who handles and knowledge of mathematics which he holds.
Perhaps the best way to see how the pantograph works is to see its bars as part of a trellis of rhombuses, the kind you can buy in garden centers for training plants on, as representing the Figure 7 (Bolt, 1991). With this image in mind it is possible to identify other ways of making linkages to produce an enlargement.

The figure 8 shows 6 ways of making a linkage to ensure that X, Y and Z are collinear (i.e. in a line) and such that YZ=2XY.

In each of these, if X is the fixed point and an object is traced out with Y, then the image traced by Z will be an enlargement from X with a linear scale factor of 3. If, however, Z traces out the object and the pencil is put at y, the image will be smaller and an enlargement from X with linear scale factor j. Note the term enlargement is still used even though the image is smaller. But the story does not end there, for we can take either Y or Z as the fixed point and in each case be left with two choices for where to put the pencil. In all the above, it should be taken into consideration the fact that the mechanistic mathematical model is able to explain its own kinematics, as a theoretically causal relation. However in other sciences, like social or partially engineering, the mathematical model can be discussed only by the function that fits the data in the best possible way, but the model fails to explain the actual meaning of the function as a theoretically causal relation. This is the reason behind the fact that mechanics are preferred for teaching mathematical modeling (Isoda & Matsuzaki, 1999).

3. Working with a Pantograph in a Secondary Classroom

In order to investigate subjects’ concepts/ theorems-in-action developed by investigating the structure of the pantograph, and especially the identification of the math concepts and laws incorporated in the machine, we selected a pantograph’s model and taught homothety to high school students for four hours (early 2016), in the framework of an attempt to incorporate artifacts with the characteristics geometrical machine’s in the instruction of Euclidean geometry. Twenty six students (16 years old), who had no prior experience with any artifact except for compasses and rulers, were asked to work with the pantograph that was a version of Scheiner’s pantograph (as Figures 9 and 10 shown). Building blocks of our the pantograph’s model were two wooden equal rods 30cm long (OD=AE=AC=BD) which were held together by the links/pivots (A,L,D,K) in the means of them forming a parallelogram (ALDK). The rods had notches allowing reassembly of the linkage while maintaining its properties provided that the links are placed in such a way that the ratios of the distances of the parts created by them are equal in each rod. The pantograph’s linkage was mounted on a wooden platform (60cmx60cm).
The tasks given to be treated by students concerned a) the explorations of the components of the structure pantograph’s (as the constrains, the length of robs, the geometrical figures, properties and relationships representing a configuration of rods, ect.), b) the investigation of the movements of the structural components of the linkage (limbs, joints, robs) to discovering the dynamic properties of the articulated system (as the finding special configurations and formations, the identifying and the analyzing invariants or changes, ect.) and c) the existing relationship between the linkage properties and the mathematical law implemented.

The purposes were the students get acquainted with linkage and begin to develop utilization schemes linked to the components; to see how they would handle the linkage’s description and what mathematics they will recognize due to the construction; to develop students utilization schemes linked to the linkage movements and provide explanations; to identify properties that are preserved or changed when the form of the machine is transformed; to define the mathematical relationship (equality ratios) that incorporates the instrument due to of its structure and operation; to correlate the relationship of the tool’s properties with the concept and properties of similarity.

4. Some Remarks

Examples of linkage give our students with very nice situations that they can explore and hopefully inquire. Simple mathematical machines in classroom activities, by reason of their characteristics and the way in which these shape and limit the possibilities of interaction with mathematical objects, regulate the production and learning of mathematical concepts. The physical and functional characteristics of the tool and the type of mathematics it incorporates, allows students to reach geometric conclusions based on the mathematics of the artifact, and may support cognitive processes focusing either on the structure of the machine, or to the embodied math concepts that emerge from the machine’s movement. Technology enables us to explore situations as Machine Engineering and Art and within mathematics itself more easier. The example of the linkage for homothety the designs that produced from the point-graph on works on paper-pencil environment are incorrect. It is not expected to emerge in a simulation within Dynamic Geometry Environment, where all the objects are drawn correct. Information Computer Technologies (ITC) are not surrogates for concrete objects; rather they have their own place in mathematics education, because of the features that are partly different from the ones of physical artifacts. In the field of Education, mathematics content itself will not change radically but the value of contents could be changed depending on what and by what means should be taught.

References


